The tangential map and associated integrable equations

V.E. Adler*

[arXiv:0906.1425] The tangential map is a map on the set of smooth planar curves. It satisfies the 3D-consistency property and is closely related to some well-known integrable equations.

* L.D. Landau Institute for Theoretical Physics

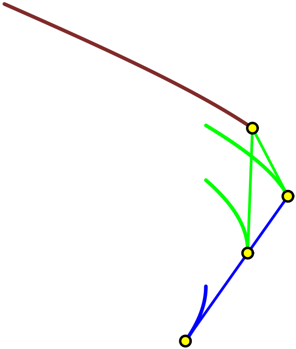
Conformal Field Theory, Integrable Models and Liouville Gravity
27.06–30.06.2009, Chernogolovka

Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet in a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined,

$$F : (C, C_1, C_2) \mapsto C_{12},$$

which will be referred to as the **tangential map**.

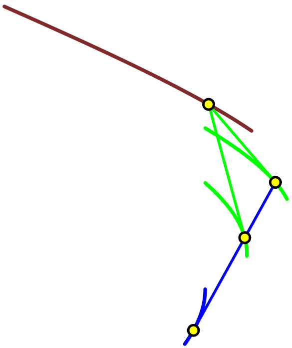


Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet in a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined,

$$F : (C, C_1, C_2) \mapsto C_{12},$$

which will be referred to as the **tangential map**.



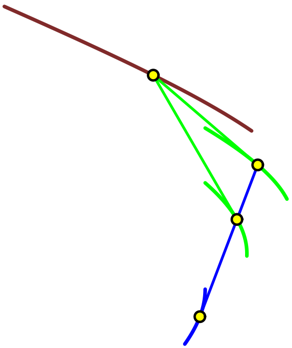
Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet in a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined,

$$F : (C, C_1, C_2) \mapsto C_{12},$$

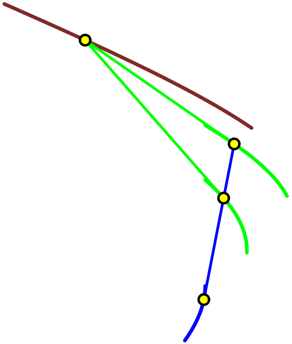
which will be referred to as the

tangential map.



Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet in a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined,



$$F : (C, C_1, C_2) \mapsto C_{12},$$

which will be referred to as the

tangential map.

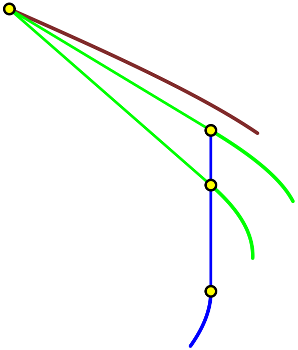
Introduction

Let smooth planar curves C, C_1 and C_2 be given. Draw the tangent line to C through any point r on this curve, and let it meet C_1 in a point r_1 and C_2 in a point r_2. Let the tangent lines to the respective curves through these points meet in a point r_{12}. When the point r moves along C, the point r_{12} draws a new curve C_{12}. Thus, a local mapping on the set of planar curves is defined,

$$F : (C, C_1, C_2) \mapsto C_{12},$$

which will be referred to as the

tangential map.



The word "local" means that:

- first, the map is defined not for all triples of curves since the tangent to C may not intersect C_1 or C_2. Only such curves or parts of the curves are considered where the construction is possible;

- second, the mapping may be multivalued since there may be several intersections. In such a case a fixed branch of the mapping is considered.

Contents of the talk

- 3D consistency property;
- relation to the factorization of differential operators;
- relation to some integrable equations;
- examples and reductions.

Example. The logarithmic spiral

The logarithmic spiral was introduced by Descartes in 1638 letter to Mersenne.

Example. The logarithmic spiral

The logarithmic spiral was introduced by Descartes in 1638 letter to Mersenne.
Jacob I Bernoulli (1654–1705)

Example. The logarithmic spiral

The logarithmic spiral was introduced by Descartes in 1638 letter to Mersenne. Jacob I Bernoulli (1654–1705) had ordered to engrave it on his tombstone[1] with the inscription

EADEM MUTATA RESURGO[2]

because this curve is invariant with respect to a lot of geometrical transformations.

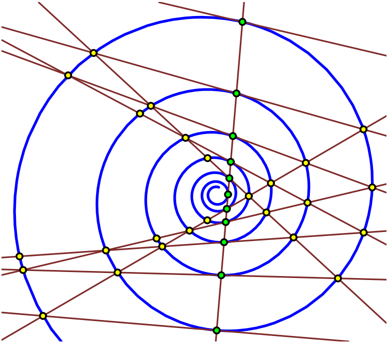
[1] The images are taken from Wikipedia
[2] I'll be back

Example. The logarithmic spiral

The logarithmic spiral was introduced by Descartes in 1638 letter to Mersenne. Jacob I Bernoulli (1654–1705) had ordered to engrave it on his tombstone[1] with the inscription

EADEM MUTATA RESURGO[2]

because this curve is invariant with respect to a lot of geometrical transformations.



It is invariant under the tangential map as well !

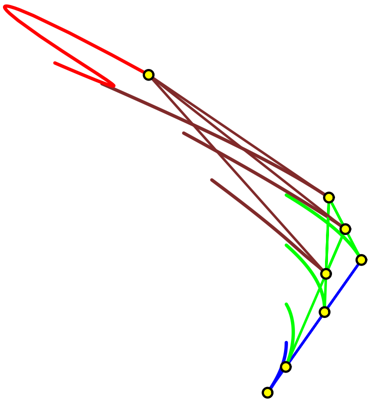
Theorem 1. Consider the intersection points of the logarithmic spiral C with its tangent. The tangents through these points meet on the same spiral. In other words, $F(C, C, C) = C$ for any branch of the map F.

[1] The images are taken from Wikipedia

[2] I'll be back

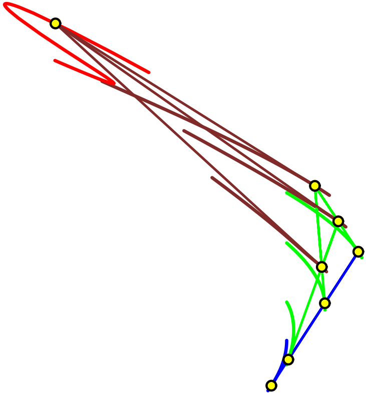
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



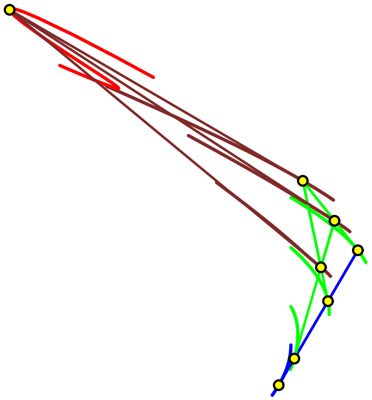
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



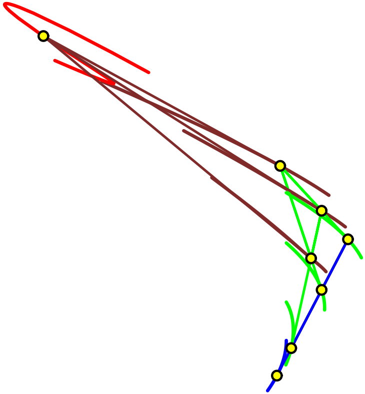
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



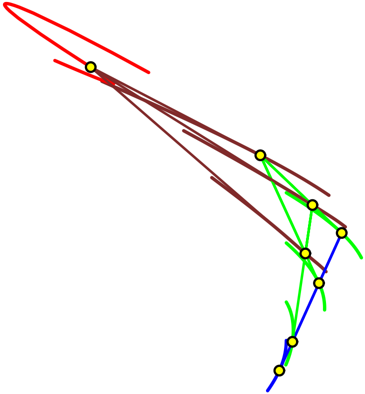
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



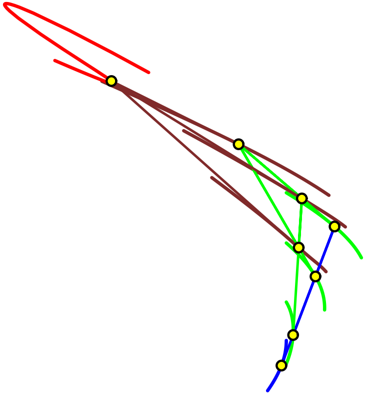
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



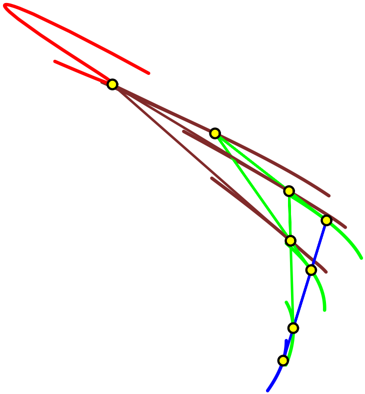
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



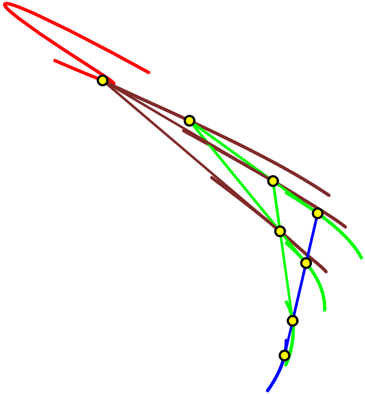
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



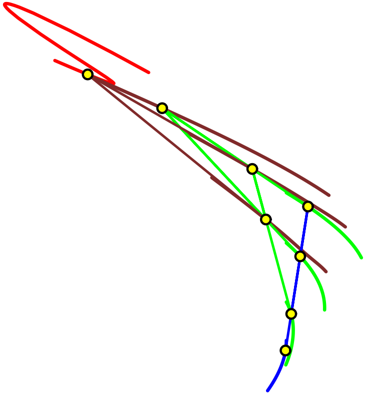
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



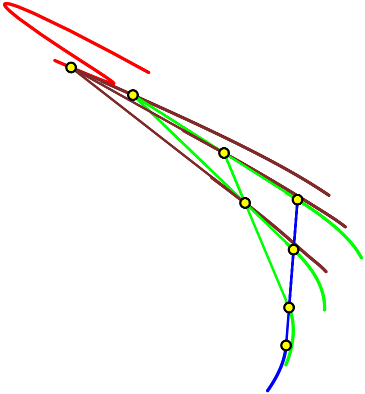
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



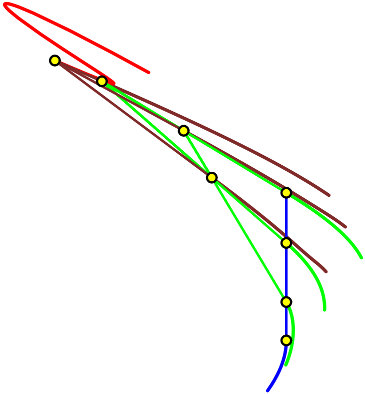
3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



3D consistency

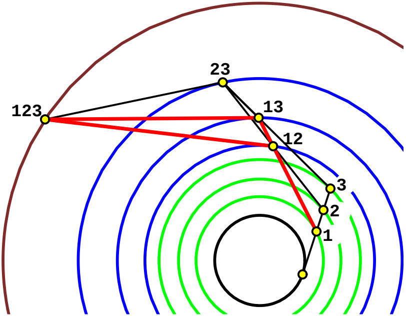
The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.

Alternatively, this can be formulated as follows.

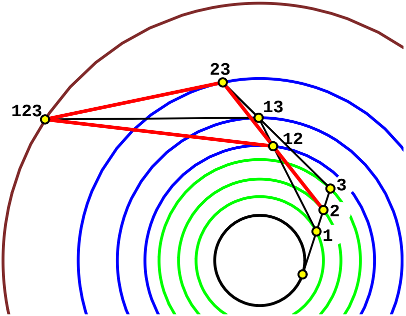


Theorem 2. The tangential map satisfies the (local) identity

$$\begin{aligned} C_{123} &= F(C_1, F(C, C_1, C_2), F(C, C_1, C_3)) \\ &= F(C_2, F(C, C_1, C_2), F(C, C_2, C_3)) \\ &= F(C_3, F(C, C_1, C_3), F(C, C_2, C_3)). \end{aligned} \tag{1}$$

3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.



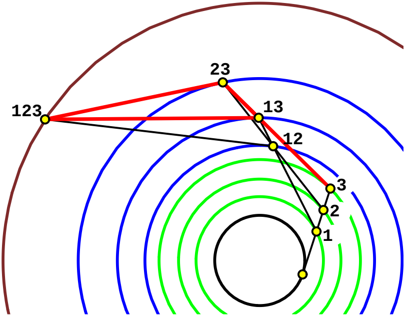
Alternatively, this can be formulated as follows.

Theorem 2. The tangential map satisfies the (local) identity

$$\begin{aligned} C_{123} &= F(C_1, F(C, C_1, C_2), F(C, C_1, C_3)) \\ &= F(C_2, F(C, C_1, C_2), F(C, C_2, C_3)) \\ &= F(C_3, F(C, C_1, C_3), F(C, C_2, C_3)). \end{aligned} \tag{1}$$

3D consistency

The main property of the tangential map is 3D-consistency. This means that if one starts from the curves C, C_1, C_2, C_3 and constructs the curves $C_{ij} = F(C, C_i, C_j)$ then the curve C_{123} constructed from the triple C_i, C_{ij}, C_{ik} is one and the same for any permutation of i, j, k.

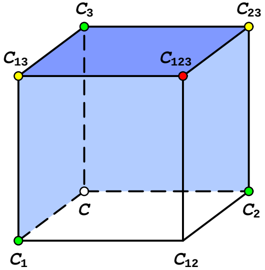


Alternatively, this can be formulated as follows.

Theorem 2. The tangential map satisfies the (local) identity

$$\begin{aligned} C_{123} &= F(C_1, F(C, C_1, C_2), F(C, C_1, C_3)) \\ &= F(C_2, F(C, C_1, C_2), F(C, C_2, C_3)) \\ &= F(C_3, F(C, C_1, C_3), F(C, C_2, C_3)). \end{aligned} \tag{1}$$

The combinatorial structure of 3D-consistency [1, 2] is represented by assigning the arguments of the map to the vertices of a cube, and the map itself to the faces. The N-fold iteration of the map is associated with an $(N+1)$-dimensional cube.



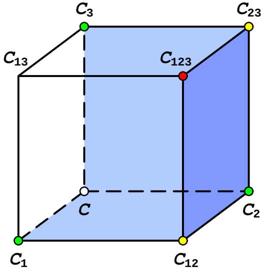
This notion is applied usually to the discrete integrable equations of the difference KdV type (the fields in the vertices of the cube) or to the Yang-Baxter type mappings [3] (the fields on the edges of the cube). These types of equations appear, for example, as nonlinear superposition principle for Darboux-Bäcklund transformations.

[1] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* ***43A*** (2001) 109–123.

[2] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notes* (2002) 573–611.

[3] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett. A* ***314:3*** (2003) 214–221.

The combinatorial structure of 3D-consistency [1, 2] is represented by assigning the arguments of the map to the vertices of a cube, and the map itself to the faces. The N-fold iteration of the map is associated with an $(N+1)$-dimensional cube.



This notion is applied usually to the discrete integrable equations of the difference KdV type (the fields in the vertices of the cube) or to the Yang-Baxter type mappings [3] (the fields on the edges of the cube). These types of equations appear, for example, as nonlinear superposition principle for Darboux-Bäcklund transformations.

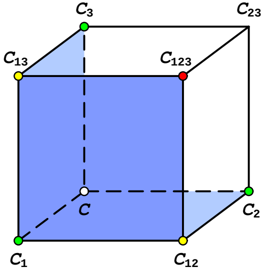
---

[1] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* ***43A*** (2001) 109–123.

[2] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notes* (2002) 573–611.

[3] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett. A* ***314:3*** (2003) 214–221.

The combinatorial structure of 3D-consistency [1, 2] is represented by assigning the arguments of the map to the vertices of a cube, and the map itself to the faces. The N-fold iteration of the map is associated with an $(N+1)$-dimensional cube.



This notion is applied usually to the discrete integrable equations of the difference KdV type (the fields in the vertices of the cube) or to the Yang-Baxter type mappings [3] (the fields on the edges of the cube). These types of equations appear, for example, as nonlinear superposition principle for Darboux-Bäcklund transformations.

[1] F.W. Nijhoff, A.J. Walker. The discrete and continuous Painlevé hierarchy and the Garnier system. *Glasgow Math. J.* **43A** (2001) 109–123.

[2] A.I. Bobenko, Yu.B. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Notes* (2002) 573–611.

[3] A.P. Veselov. Yang-Baxter maps and integrable dynamics. *Phys. Lett. A* **314:3** (2003) 214–221.

The simplest examples

- Fields in the vertices:

Discrete wave equation

$$u + u_i + u_j + u_{ij} = 0$$

Independently on the order of computation:

$$u_{123} = 2u + u_1 + u_2 + u_3.$$

The simplest examples

- Fields in the vertices:

Bianchi permutability theorem for Korteweg-de Vries equation

$$(u - u_{ij})(u_i - u_j) = \mu^j - \mu^i$$

Independently on the order of computation:

$$u_{123} = -\frac{(\mu^2 - \mu^1)u_1 u_2 + c.p.}{(\mu^2 - \mu^1)u_3 + c.p.}$$

The simplest examples

- Fields in the vertices:

Bianchi permutability theorem for sinh-Gordon (Hirota eq [4])

$$\mu^i(uu_i + u_j u_{ij}) = \mu^j(uu_j + u_i u_{ij}) \tag{2}$$

Independently on the order of computation:

$$u_{123} = -\frac{((\mu^2)^2 - (\mu^1)^2)\mu^3 u_1 u_2 + c.p.}{((\mu^2)^2 - (\mu^1)^2)\mu^3 u_3 + c.p.}$$

[4] R. Hirota. Nonlinear partial difference equations. III. Discrete sine-Gordon equation. *J. Phys. Soc. Japan* ***43*** (1977) 2079–2086.

The simplest examples

- Fields in the vertices:

Bianchi permutability theorem for sinh-Gordon (Hirota eq [4])

$$\mu^i(uu_i + u_j u_{ij}) = \mu^j(uu_j + u_i u_{ij}) \tag{2}$$

Independently on the order of computation:

$$u_{123} = -\frac{((\mu^2)^2 - (\mu^1)^2)\mu^3 u_1 u_2 + c.p.}{((\mu^2)^2 - (\mu^1)^2)\mu^3 u_3 + c.p.}$$

- Fields on the edges:

another form of discrete KdV ($a^j = u_j - u$):

$$a_i^j = a^i + \frac{\mu^i - \mu^j}{a^i - a^j}$$

[4] R. Hirota. Nonlinear partial difference equations. III. Discrete sine-Gordon equation. *J. Phys. Soc. Japan* ***43*** (1977) 2079–2086.

Important distinctions in our case:

- Both dKdV or YB types of equations are 2D: two discrete independent variables correspond to the shifts along the edges of an elementary square. In contrast, the tangential map is related to a 3D equation: in addition to the discrete variables a continuous one appears corresponding to a parameter along the curves.

- The tangential map is asymmetric: the roles of the involved curves are obviously different. In particular, we will see that the construction of C_{ij} from C, C_i, C_j is described by a differential mapping, while the construction, for instance, of C_j from C, C_i, C_{ij} requires an integration.

In the terminology of [5], the non-degeneracy condition is not fulfilled. This explains the choice of the set C, C_i, C_j, $C_k, \dots$ as preferable initial data, rather that the sequence C, C_i, C_{ij}, $C_{ijk}, \dots$, as it is usual in the standard formulation of YB mappings.

---

[5] P. Etingof, T. Schedler, A. Soloviev. Set-theoretical solutions to the quantum Yang-Baxter equation. *Duke* ***100:2*** (1999) 169–209.

Factorization of differential operators

Let the curve C be given in a parametric form $r = r(t)$. The intersection with the curve C_i is given by an equation of the form

$$r_i(t) = r(t) + a^i(t)\dot{r}(t), \qquad \dot{r} := D(r) := \frac{dr}{dt}$$

which plays the role of an auxiliary linear problem for one branch of the tangential map. The curve C_{ij} is defined from the compatibility condition

$$(1 + a^j_i D)(1 + a^i D) = (1 + a^i_j D)(1 + a^j D), \tag{3}$$

where the coefficient a^i_j correspond to the edge $C_j C_{ij}$. This equation can be solved as the differential mapping

$$(a^i, a^j) \mapsto (a^i_j, a^j_i), \qquad \boxed{\boldsymbol{a^i_j = \frac{(a^i - a^j)a^i}{a^i - a^j + a^i\dot{a}^j - \dot{a}^i a^j}}}. \tag{4}$$

Alternatively, the potential v defined accordingly to the formula $a^i = v/v_i$ is governed by the map

$$f:(v,v_i,v_j)\mapsto v_{ij},\qquad \boldsymbol{v_{ij}=\frac{v_iv_j}{v}+\frac{\dot v_iv_j-v_i\dot v_j}{v_j-v_i}}. \tag{5}$$

Mappings (4), (5) are interpreted as 3-dimensional equations on $\mathbb{Z}^2\times\mathbb{R}$, with the fields a corresponding to the edges of the lattice and v corresponding to the vertices. These equations are related via simple substitutions to the semidiscrete Toda lattice, introduced in [6] for the first time (to the best of author's knowledge), see also [7].

- A bit more simple mappings are obtained under the normalization with the unitary leading term:

$$(D-a^j_i)(D-a_i)=(D-a^i_j)(D-a_j).$$

---

[6] D. Levi, L. Pilloni, P.M. Santini. Integrable three-dimensional lattices. *J. Phys. A* ***14:7*** (1981) 1567–1575.

[7] V.E. Adler, S.Ya. Startsev. Discrete analogues of the Liouville equation. *Theor. Math. Phys.* ***121:2*** (1999) 271–284.

The property of 3D-consistency is equivalent to the commutativity of the operators $T_i : a^j \to a_i^j$:

$$T_i T_j (a^k) = T_j T_i (a^k), \tag{6}$$

or to the identity of type (1):

$$\begin{aligned} v_{123} &= f(v_1, f(v, v_1, v_2), f(v, v_1, v_3)) \\ &= f(v_2, f(v, v_1, v_2), f(v, v_2, v_3)) \\ &= f(v_3, f(v, v_1, v_3), f(v, v_2, v_3)). \end{aligned} \tag{7}$$

Both identities can be proved straightforwardly, although the computation is rather tedious. It can be avoided by the following argument.

Proof of Theorem 2. Accordingly to (3), the tangential map amounts to the reconstruction of an ordinary second order differential operator from its kernel, under the condition of unitary constant term. Consider the differential operator

$$A = (1 + T_i(a^k_j)D)(1 + a^j_i D)(1 + a^i D)$$

corresponding to one of three possible ways of computing r_{ijk}.

Eq (3) implies that A is divisible from the right not only by $1 + a^i D$, but also by $1 + a^j D$. Moreover, two left factors of A can be rewritten as $(1 + T_i(a^j_k)D)(1 + a^k_i D)$, that is operator A does not changes under permutation of j and k. But this means that it is divisible from the right by $1 + a^k D$ as well.

The invariance of the kernel with respect to any permutation of indices implies the invariance of the operator A itself. ■

The N-fold tangential map corresponds to a differential operator of order N divisible from the right by operators $1+a^i D$, $i = 1, \ldots, N$. This immediately leads to the Wronskian formula (for each of two components of r)

$$
r_{1,2,\ldots,N} = \frac{\det \begin{pmatrix} r & \varphi_1 & \varphi_2 & \ldots & \varphi_N \\ \dot{r} & \dot{\varphi}_1 & \dot{\varphi}_2 & \ldots & \dot{\varphi}_N \\ \vdots & \vdots & \vdots & & \vdots \\ D^N(r) & D^N(\varphi_1) & D^N(\varphi_2) & \ldots & D^N(\varphi_N) \end{pmatrix}}{\det \begin{pmatrix} \dot{\varphi}_1 & \dot{\varphi}_2 & \ldots & \dot{\varphi}_N \\ \vdots & \vdots & & \vdots \\ D^N(\varphi_1) & D^N(\varphi_2) & \ldots & D^N(\varphi_N) \end{pmatrix}}
$$

where $a^i = -\varphi_i/\dot{\varphi}_i$.

Example. Logarithmic spirals and concentric circles

Let $r = e^{(\gamma+\mathrm{i})t}$ ($\gamma = 0$ correspond to the circle). Then the curves

$$r_k = r + a^k \dot{r} = (1 + \gamma a^k + \mathrm{i}a^k)e^{(\gamma+\mathrm{i})t}$$

are homothetic to the original one if and only if the coefficients a^k are constant. The action of the map (4) on the constant coefficients is identical: $a^k_j = a^k$, therefore the tangential map amounts to the rotational dilation

$$r_{jk} = (1 + \gamma a^j + \mathrm{i}a^j)(1 + \gamma a^k + \mathrm{i}a^k)r$$

which preserves the family of curves under consideration. The N-fold mapping is given by analogous explicit formula, so that this example can be considered trivial.

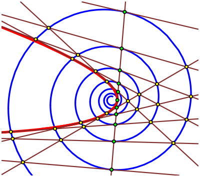
- Even this simplest example shows that the correspondence between the tangential map and differential operators is not one-to-one. It depends on the choice of initial curve and its parametrization. The mappings corresponding to the same operators are locally equivalent, but the global picture may be quite different. For example, the number of branches is 4 the case of concentric circles and ∞ in the case of logarithmic spirals.

- The auto-map of log spiral is obtained under the additional constraint $r_k(t) = r(t + \delta_k)$ This implies that δ_k are roots of equation

$$\cos\delta - \gamma\sin\delta = \exp(-\gamma\delta),$$

and the coefficients a^k are expressed by the formula

$$a^k = \exp(\gamma\delta_k)\sin\delta_k.$$

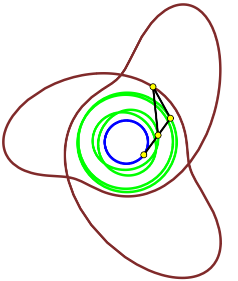


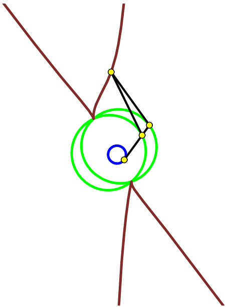
It can be deduced from here that the boundary of the domain free of the lines is approximated by a parabola.

Example. Periodic coefficients

A picture with good global behavior of the curves can be obtained if the starting curve is the circle $r = e^{\mathrm{i}t}$ again, and the coefficients $a^k(t)$ are periodic with periods commensurable with π. For example, the plots below correspond to the coefficients of the general form

$$a^k = \alpha + \beta \sin\left(\frac{m}{n} t + \gamma\right).$$

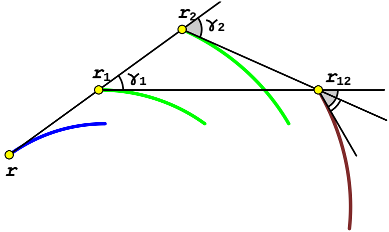




Reduction. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a **loxodrome** for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an **involute** of C).

Remarkably, the tangential map preserves this type of relation between the curves.

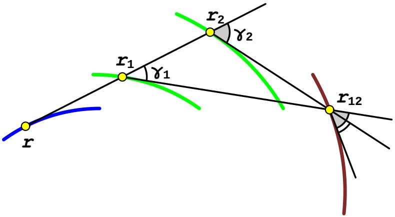


Theorem 3. Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.

Reduction. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a **loxodrome** for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an **involute** of C).

Remarkably, the tangential map preserves this type of relation between the curves.

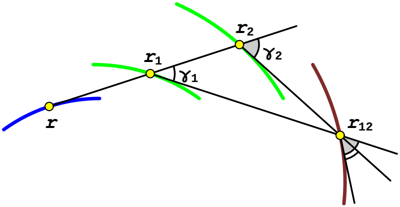


Theorem 3. Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.

Reduction. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a **loxodrome** for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an **involute** of C).

Remarkably, the tangential map preserves this type of relation between the curves.

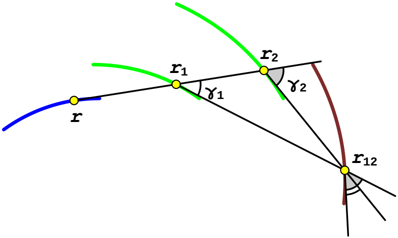


Theorem 3. Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.

Reduction. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a **loxodrome** for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an **involute** of C).

Remarkably, the tangential map preserves this type of relation between the curves.

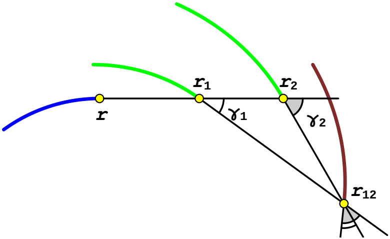


Theorem 3. Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.

Reduction. Loxodromes

We will say, slightly abusing the terminology, that a curve $\widetilde{C}$ is a **loxodrome** for a given curve C if it intersects the tangents to C under a constant angle γ (in particular, if $\gamma = \pi/2$, then $\widetilde{C}$ is an **involute** of C).

Remarkably, the tangential map preserves this type of relation between the curves.

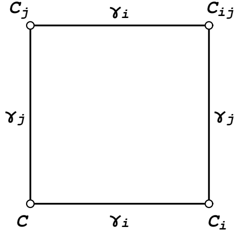


Theorem 3. Let curves C_i and C_j meet tangents to a curve C under constant angles γ^i and γ^j respectively, and $\gamma^i \neq \gamma^j$. Then the curve $C_{ij} = F(C, C_i, C_j)$ meets tangents to C_i under the angle γ^j and tangents to C_j under the angle γ^i.

• Formally, this statement looks like Bianchi theorem on the permutability of Darboux-Bäcklund type transformations. However, the situation in this case turns out to be more simple: the construction of the loxodromes amounts to a simple quadrature, while Bäcklund transformations amount to solving of Riccati equations.

The superposition principle for the transformations under consideration can be brought to the linear form:

$$\sin(\gamma^i - \gamma^j)y_{ij} = \sin(\gamma^i)y_i - \sin(\gamma^j)y_j.$$

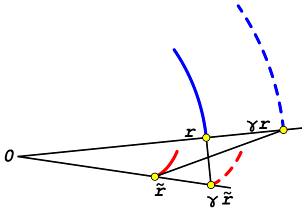


A genuine Darboux transformation leading to the nonlinear superposition principle is provided by the reduction presented in the next example.

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



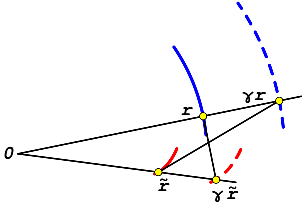
Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i \gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



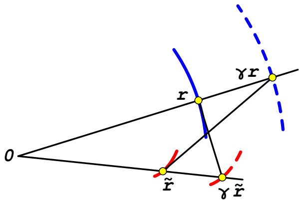
Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i\gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



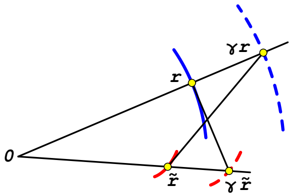
Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i\gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



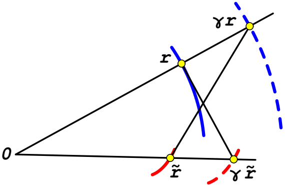
Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i\gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



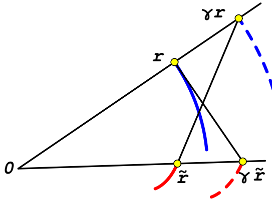
Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i \gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

Reduction. Darboux transformation

Let H_γ be the homothety with a coefficient $\gamma \neq 1$ with respect to a fixed point O. The curves C, $\widetilde{C}$ are in the **tangential correspondence** with parameter γ if the tangent to C through any point r meets $H_\gamma(\widetilde{C})$ in the point $H_\gamma(\tilde{r})$, and the tangent to $\widetilde{C}$ through $\tilde{r}$ meets $H_\gamma(C)$ in the point $H_\gamma(r)$.

This constraint is preserved under the tangential map as well.



Theorem 4. Let a curve C be in the tangential correspondence with curves C_i, C_j, with parameters γ^i, γ^j respectively, and $\gamma^i \neq \gamma^j$. Then an unique curve C_{ij} exists which is in the tangential correspondence with the curves C_i, C_j, with the parameters γ^j, γ^i respectively. Moreover,

$$H_{\gamma^i\gamma^j}(C_{ij}) = F(C, H_{\gamma^i}(C_i), H_{\gamma^j}(C_j)).$$

• It can be proved that, under a suitable parametrization, the tangential correspondence is equivalent to the Darboux transformation for the equation

$$\ddot{r} + u\dot{r} + \lambda r = 0$$

which is nothing but the spectral problem for the $\sinh$-Gordon equation. The mappings (4), (5) take the algebraic forms

$$a^i_j = \frac{a^i - a^j}{a^j(\mu^i a^i - \mu^j a^j)}, \qquad v_{ij}(v_j - v_i) = v(\mu_i v_j - \mu_j v_i)$$

equivalent respectively to the mapping (F_4) from [8] and to the Hirota equation (2). These equations express the nonlinear superposition principle for the Darboux transformation.

[8] V.E. Adler, A.I. Bobenko, Yu.B. Suris. Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. *Comm. Anal. and Geom.* ***12:5*** (2004) 967–1007.

Discrete tangential map

An analog of the tangential map for the discrete curves $r = r(n)$ reads

$$r_i = r + a^i(T-1)(r), \quad T : n \mapsto n+1.$$

This leads to the factorization of the difference operators:

$$(1 + a^i_j(T-1))(1 + a^j(T-1)) = (1 + a^j_i(T-1))(1 + a^i(T-1))$$

and to the mappings ($a^i = T(v)/v_i$)

$$(a^i, a^j) \mapsto (a^i_j, a^j_i), \quad \boldsymbol{a^i_j = \frac{(a^i - a^j)T(a^i)}{(1-a^j)T(a^i) - (1-a^i)T(a^j)}}, \tag{8}$$

$$f : (v, v_i, v_j) \mapsto v_{ij}, \quad \boldsymbol{v_{ij} = \frac{v_i v_j T(v_j - v_i)}{T(v)(v_j - v_i)} + \frac{T(v_i)v_j - v_i T(v_j)}{v_j - v_i}}. \tag{9}$$

- The symmetric form of (9)

$$\frac{T(v_j - v_i)}{T(v)} + \frac{T(v_i) - v_{ij}}{v_i} + \frac{v_{ij} - T(v_j)}{v_j} = 0,$$

shows that the shift T is actually on the equal footing with T_i and T_j.

- Difference substitutions relate this equation to the discrete Toda and KP equations (in particular, the variable v is identified as the wave function of the linear problem for KP equation [9]). Alternative geometric interpretations of these equations can be found in the papers [10, 9], see also [11] where a general theory of this class of equations is developed.

---

[9] B.G. Konopelchenko, W.K. Schief. Menelaus' theorem, Clifford configurations and inversive geometry of the Schwarzian KP hierarchy. *J. Phys. A* ***35:29*** (2002) 6125–6144.

[10] A. Doliwa. Geometric discretization of the Toda system. *Phys. Lett. A* ***234*** (1997) 187–192.

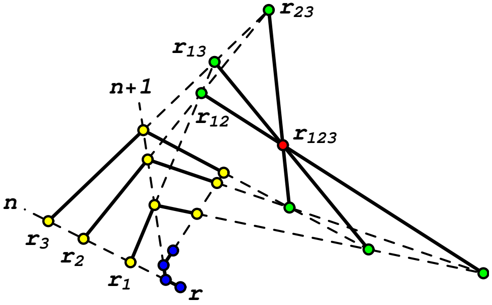
[11] V.E. Adler, A.I. Bobenko, Yu.B. Suris. The classification of integrable discrete equations of octahedron type, to appear, 2009.

- The 3D-consistency property of the maps (8) and (9) is formulated by the same general identities (6), (7) as in the continuous case, and is proved along the lines of the proof of Theorem 2.

- There exists also the simple geometric explanation of this property[3]: the triangles $r_{12}(n)r_{13}(n)r_{23}(n)$ and $r_{12}(n+1)r_{13}(n+1)r_{23}(n+1)$ are perspective with respect to the line $r(n+1)r(n+2)$ (marked by $n+1$ on the figure), therefore, accordingly to Desargues theorem, the lines

$$r_{12}(n)r_{12}(n+1),$$
$$r_{13}(n)r_{13}(n+1),$$
$$r_{23}(n)r_{23}(n+1)$$

are concurrent, as required.



[3] This proof is due to W.K. Schief

Conclusion

The tangential mapping is not a quite new object, rather it is of certain interest as one more geometric interpretation of well-known integrable equations, 3D and 2D (under some reductions):

- semidiscrete Toda lattice ($\Delta\Delta D$);
- Hirota equation ($\Delta\Delta$, under a reduction);
- a modification of discrete KP equation ($\Delta\Delta\Delta$, in the discrete version of the map).